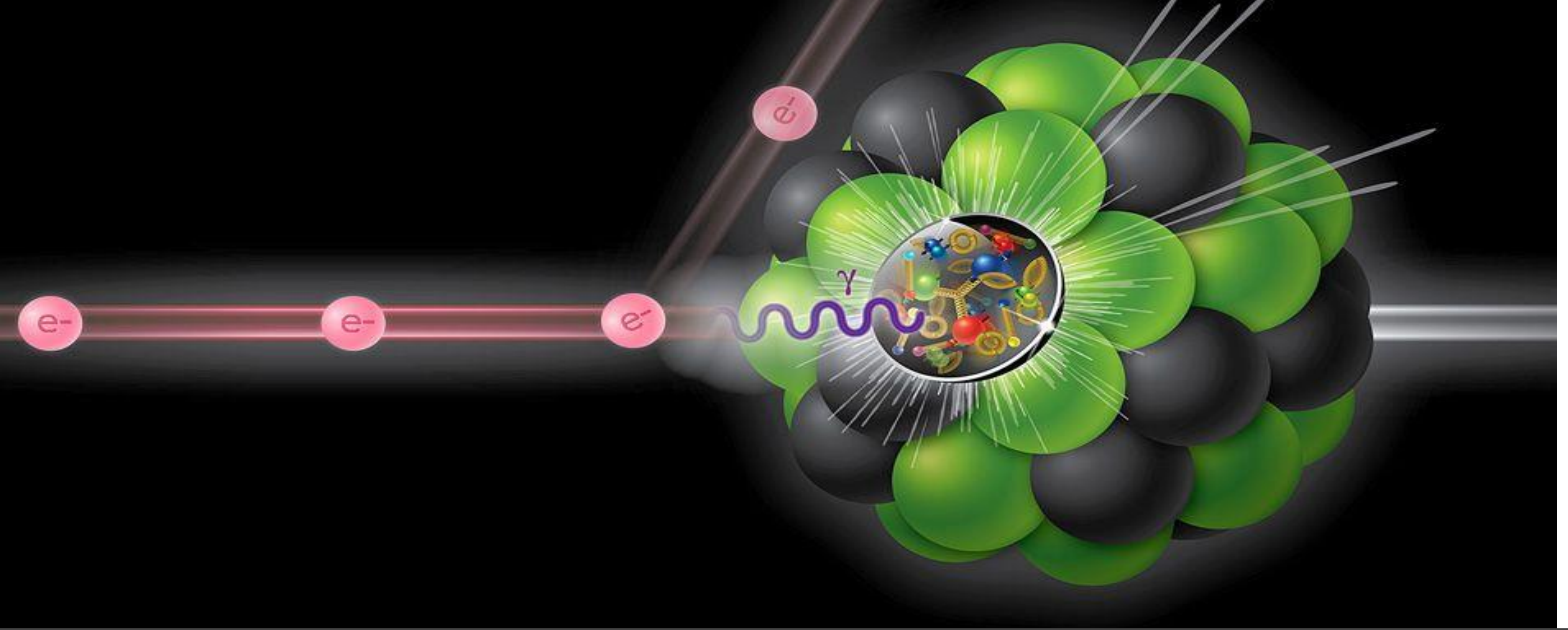

Particle Physics I

Lecture 4: Particle decay rates

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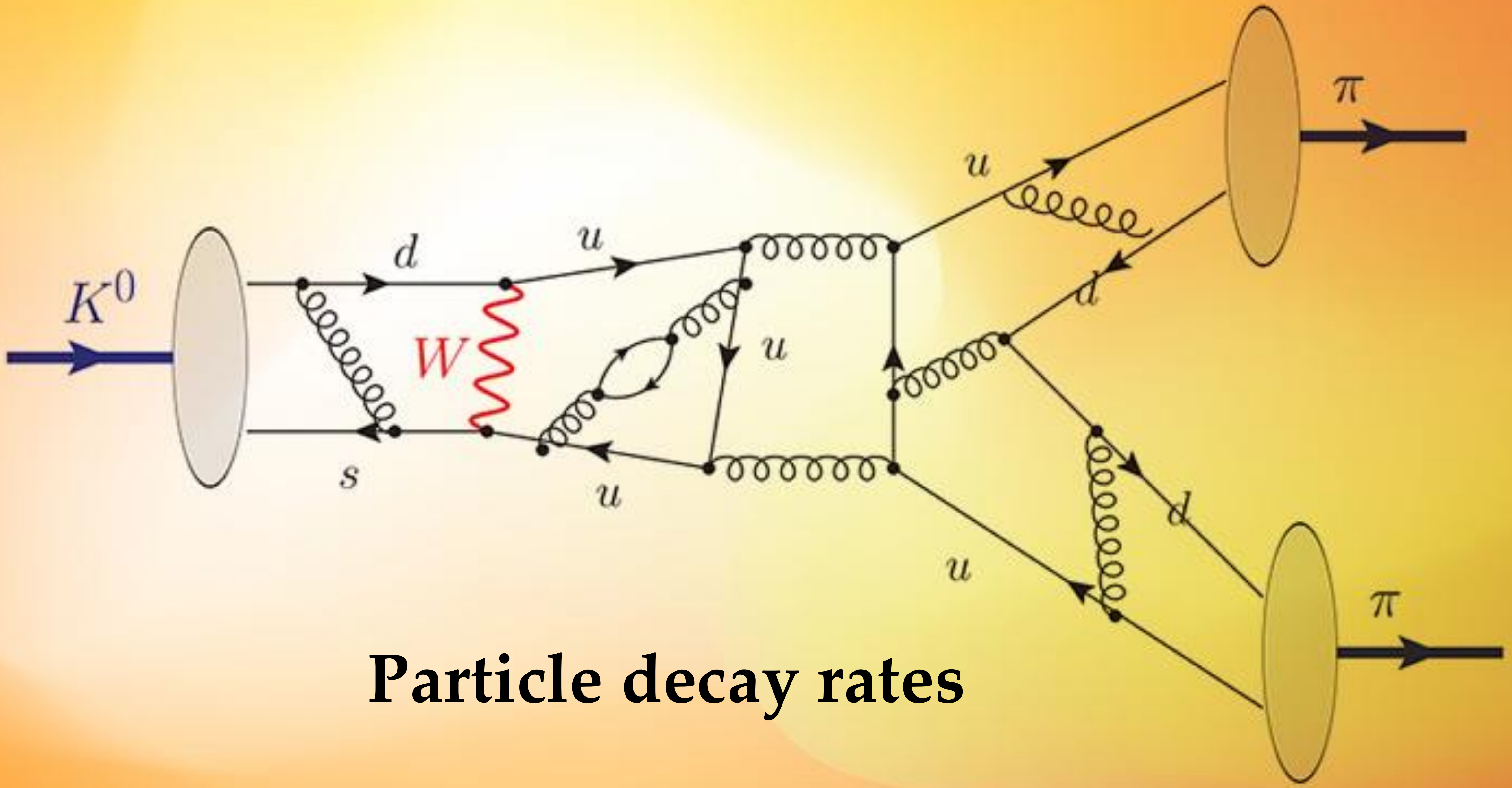
October 1st 2025

Today's learning targets

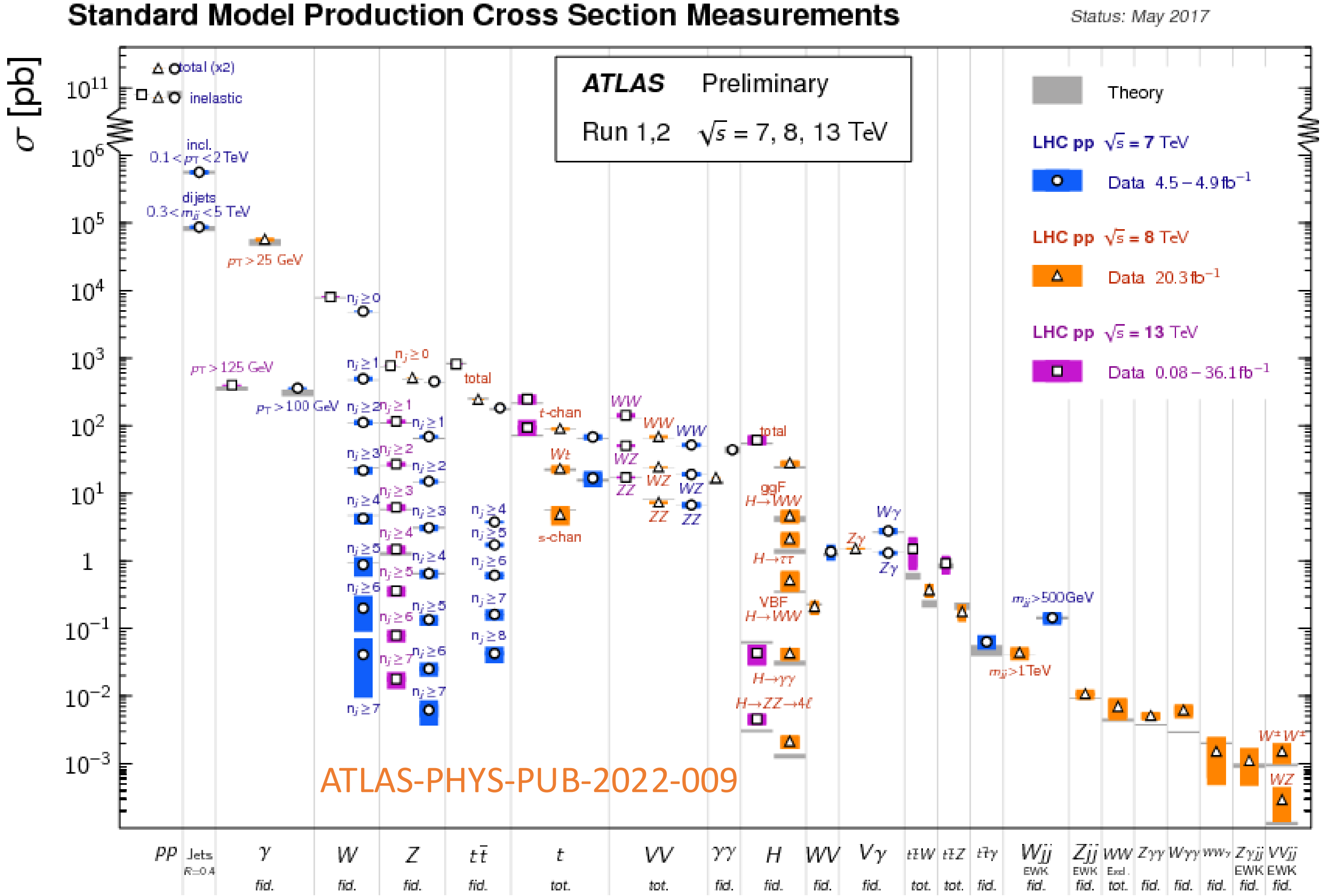
- How to compute particle decay rates
- What is Fermi's golden rule
- How to compute 2-body decay rates using Fermi's golden rule



Particle decay rates

Cross sections and decay rates

- All calculation in particle physics revolve around particle **interactions** and **decays** (**transition between states**)



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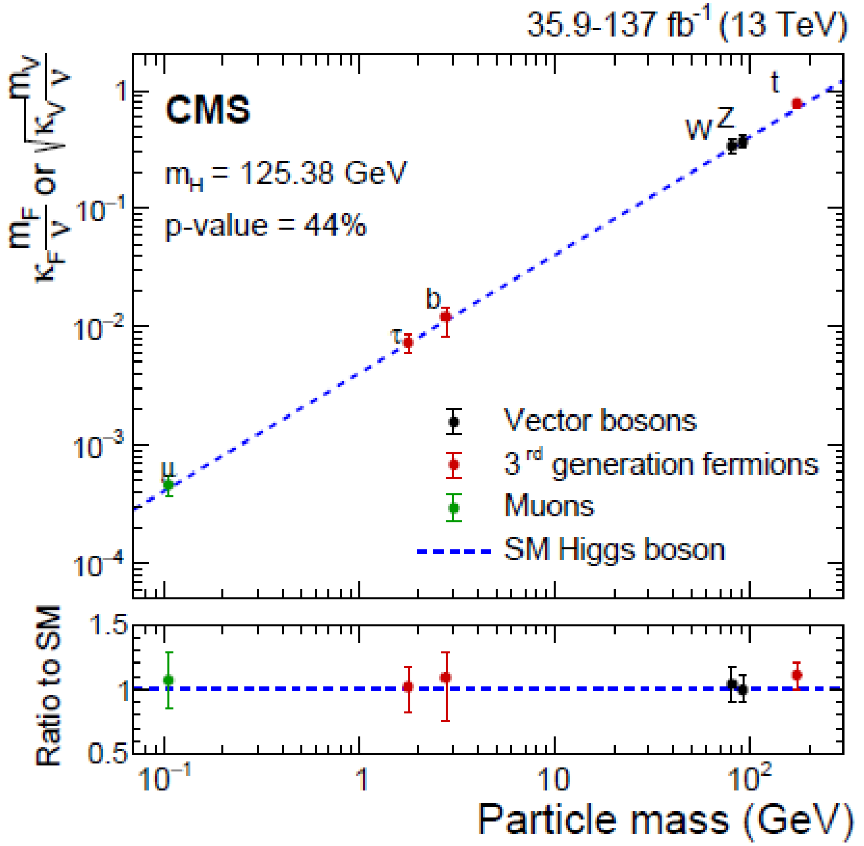

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Cross sections and decay rates

- We can calculate transition rates using Fermi's Golden Rule (see Chapter 2.3.6 in Thomson for the derivation) :

$$\Gamma_{fi} = 2\pi \left|T_{fi}\right|^2 \rho(E_f) \qquad (1)$$

- **Γ_{fi}**: number of transitions per unit time from initial state $|i\rangle$ to final state $|f\rangle$ **(not Lorentz Invariant!)**

- **T_{fi}**: transition matrix element (ME) determined by the Hamiltonian of the interaction (j is an intermediate particle)

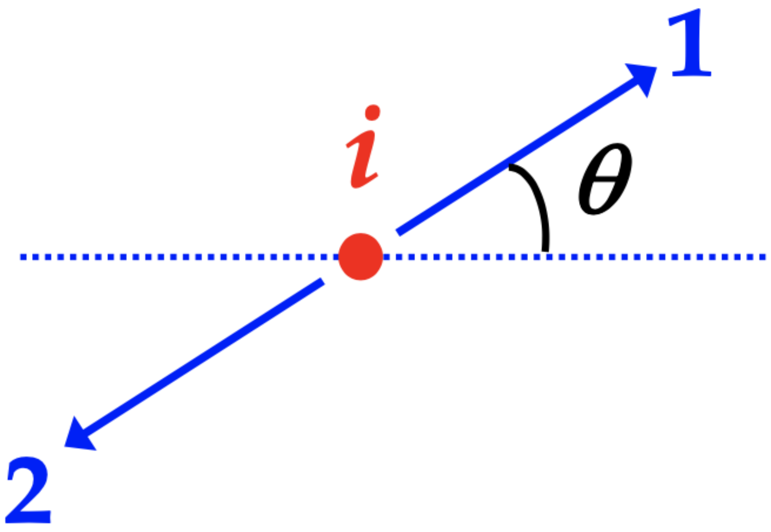
weak perturbation

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j\neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \ \ldots \qquad (2)$$

- **$\rho(E_f)$**: density of final states, $\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f}$

- Decay rates depend on **matrix elements** (= fundamental particle physics) and **densities of states** (= kinematics)

Particle decay rates



- Two-body decay $i \rightarrow 1 + 2$

- The transition matrix element in first order perturbation theory is given by:

$$T_{fi} = \langle \Psi_1 \Psi_2 | \hat{H} | \Psi_i \rangle = \int_V \Psi_1^* \Psi_2^* \hat{H} \Psi_i \, d^3x \tag{3}$$

- Calculate the decay rate in first order perturbation theory describing the particle motion using plane-wave (Born approximation)

$$\Psi_1 = N e^{-i(\vec{p}.\vec{r} - Et)} = N e^{-ip.x} \tag{4}$$

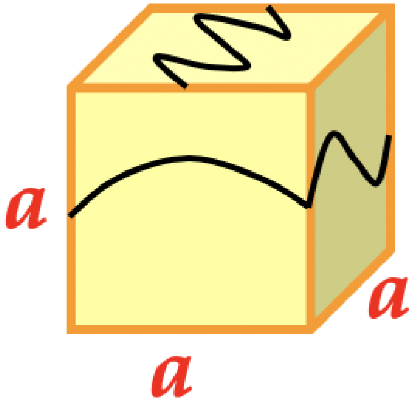
N is the normalisation

Particle decay rates

- For the decay-rate computation we need to know (in a Lorentz Invariant form)
  - wave-function normalisation
  - transition matrix element from perturbation theory
  - expression for the density of states
- Let's consider wave-function normalisation first:
  - non-relativistic formulation: normalise to one particle per cube of size a

$$\int \Psi\Psi^* \, dV = N^2 a^3 = 1 \implies N^2 = 1/a^3 \qquad (5)$$

Non-relativistic phase space

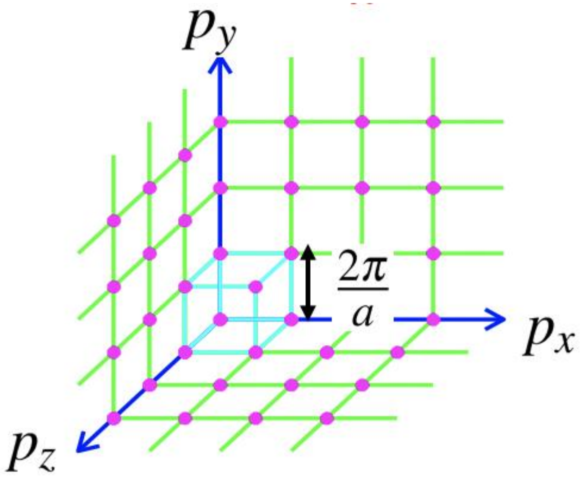


- Apply boundary conditions: $\vec{p} = \hbar\vec{k}$

- Periodic boundary conditions on the wave function:

  - $\Psi(x+a, y, z) = \Psi(x, y, z) \implies$ quantized particle momentum

$$p_x = \frac{2\pi n_x}{a}; \qquad p_y = \frac{2\pi n_y}{a}; \qquad p_z = \frac{2\pi n_z}{a} \tag{6}$$



- Volume of single state in momentum space:

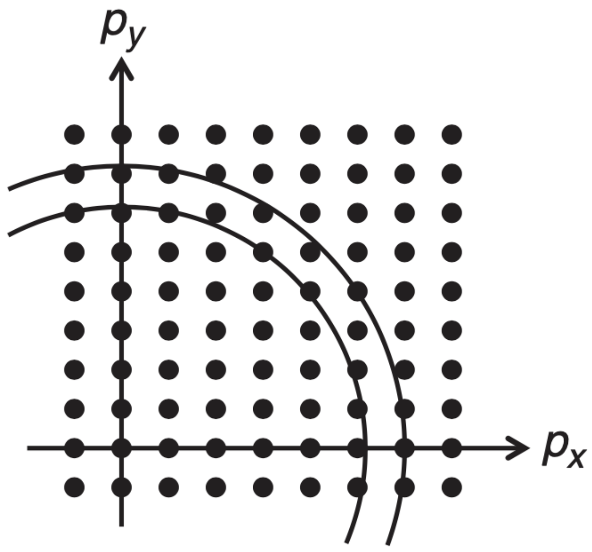
$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V} \tag{7}$$

- Normalising to one particle per unit volume gives the number of states in an element

$d^3\vec{p} = dp_x dp_y dp_z$

$$dn = \frac{d^3\vec{p}}{(2\pi)^3/V} \times \frac{1}{V} = \frac{d^3\vec{p}}{(2\pi)^3} \tag{8}$$

Non-relativistic phase space



- Density of states in the Golden rule: $\vec{p} = \hbar \vec{k}$

$$\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f} = \left|\frac{dn}{d|\vec{p}|}\frac{d|\vec{p}|}{dE}\right|_{E_f} \qquad \textbf{(9)}$$

- Transformation of the elements using Eq. 8 and 9: $d^3\vec{p} = 4\pi p^2 dp$ (spherical coordinates)

$$\frac{dn}{d|\vec{p}|} = \frac{1}{(2\pi)^3}\frac{d^3\vec{p}}{d|p|} = \frac{4\pi p^2 dp}{(2\pi)^3 dp} = \frac{4\pi p^2}{(2\pi)^3} \qquad \textbf{(10)}$$

$$E^2 = p^2 + m^2 \implies 2EdE = 2pdp \implies \frac{dp}{dE} = \frac{E}{p} \approx \frac{1}{\beta} \qquad \textbf{(11)}$$

$$\implies \rho(E_f) = \frac{4\pi p^2}{(2\pi)^3}\frac{1}{\beta} \qquad \textbf{(12)}$$

- Larger final state momentum implies larger density of states (all other things being equal)
  - decays to lighter particles are preferred

The Golden rule revisited

$$\Gamma_{fi} = 2\pi \left|T_{fi}\right|^2 \rho\left(E_f\right) \qquad \mathbf{(13)}$$

- Rewrite the expression for density of states using a Dirac`s δ −function (backup slides?)
- Transformation of the elements using Eq. 8 and 9: $d^3\vec{p} = 4\pi p^2 dp$ (spherical coordinates)

$$\rho\left(E_f\right) = \left|\frac{dn}{dE}\right|_{E_f} = \int \frac{dn}{dE}\delta(E - E_i)dE \quad \text{since } E_f = E_i \qquad \mathbf{(14)}$$

- Note: integrating over all final states energies but energy conservation now taken into account explicitly by the δ −function
- Hence the golden rule becomes an integral over all “allowed” final states of **any energy**:

$$\Gamma_{fi} = 2\pi \int \left|T_{fi}\right|^2 \delta(E_i - E)dn \qquad \mathbf{(15)}$$

The Golden rule revisited

- For dn in a two-body decay, we only need to consider one particle as **momentum conservation fixes the other**

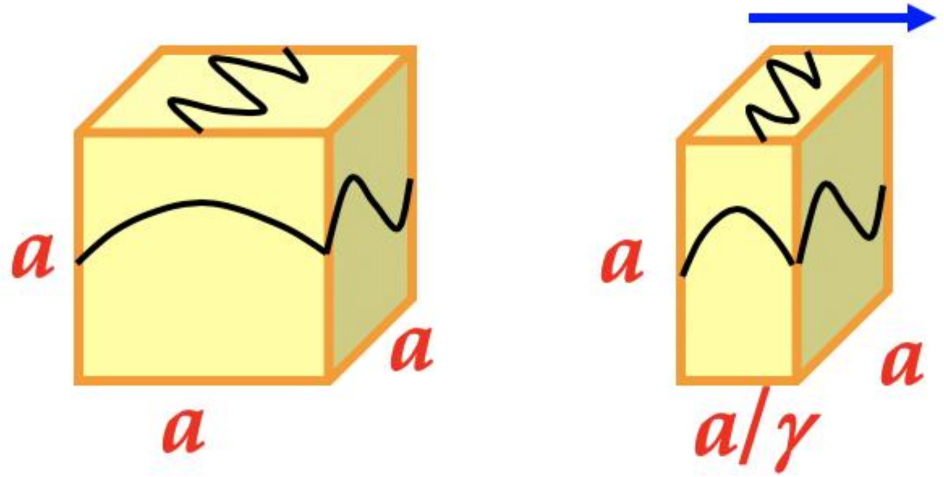
$$\Gamma_{fi} = 2\pi \int \left|T_{fi}\right|^2 \delta(E_i - E_1 - E_2) \frac{d^3\overrightarrow{p_1}}{(2\pi)^3} \qquad \textbf{(16)}$$

- We can also include momentum conservation explicitly by integrating over the momenta of both particles and using another δ –function

$$\Gamma_{fi} = (2\pi)^4 \int \left|T_{fi}\right|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{E\text{ conservation}} \underbrace{\delta^3(\overrightarrow{p_i} - \overrightarrow{p_1} - \overrightarrow{p_2})}_{\vec{p}\text{ conservation}} \underbrace{\frac{d^3\overrightarrow{p_1}}{(2\pi)^3} \frac{d^3\overrightarrow{p_2}}{(2\pi)^3}}_{\text{density of states}} \qquad \textbf{(17)}$$

Lorentz invariant phase space

- In non-relativistic QM normalise to one particle/unit volume: $\int \Psi^* \Psi dV = 1$
- Considering relativistic effects: moving to different reference frame, volume contracts by $\gamma = E/m$



- Particle density must therefore increase by γ $\Longrightarrow$ Lorentz invariant wave-function normalisation must be proportional to E particles per unit volume

Lorentz invariant phase space

- Usual convention: normalise to $2E$ particles per unit volume: $\int \Psi'^{*}\Psi' dV = 2E$
- $\Psi' = \sqrt{2E}\Psi$ is properly normalised to take into account relativistic space-time contraction
- Define Lorentz invariant matrix element, M_{fi}, in terms of the wave–functions normalized to $2E$ particles per unit volume:

$$\boldsymbol{M_{fi}} = \langle \underbrace{\boldsymbol{\Psi_1'\Psi_2'} \ldots}_{\text{final state}} \left| \widehat{\boldsymbol{H}} \right| \underbrace{\boldsymbol{\Psi_a'\Psi_b'} \ldots}_{\text{initial state}} \rangle = \sqrt{\boldsymbol{2E_1 2E_2 \ldots 2E_a 2E_b}} \times \boldsymbol{T_{fi}} \qquad \textbf{(18)}$$

Two-body decay

$$M_{fi} = \langle \Psi_1' \Psi_2' | \hat{H} | \Psi_i' \rangle = \sqrt{2E_1 2E_2 2E_i} \times \langle \Psi_1 \Psi_2 | \hat{H} | \Psi_\mathrm{i} \rangle = \sqrt{2E_1 2E_2 2E_i} \times T_{fi} \qquad \mathbf{(19)}$$

- Expressing T_{fi} in terms of M_{fi} then gives

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\overrightarrow{p_i} - \overrightarrow{p_1} - \overrightarrow{p_2}) \frac{d^3\overrightarrow{p_1}}{(2\pi)^3 2E_1} \frac{d^3\overrightarrow{p_2}}{(2\pi)^3 2E_2} \qquad \mathbf{(20)}$$

- M_{fi} uses relativistically normalised wave-functions and is **Lorentz invariant**
- $\frac{d^3\vec{p}}{(2\pi)^3 2E}$ is the Lorentz invariant phase space for each final state particle
  - the factor of $2E$ arises from the wave-function normalisation

Two-body decay

- This form of Γ_{fi} is simply a rearrangement of the original equation but the integral is now frame-independent (Lorentz invariant)
- Γ_{fi} is inversely proportional to E_i, the energy of the decaying particle
  - this is an expected effect induced by time dilation
- Energy and momentum conservation are explicitly imposed by the δ −functions

Decay rate calculation

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^3(\vec{p_i} - \vec{p_1} - \vec{p_2}) \frac{d^3\vec{p_1}}{(2\pi)^3 2E_1}\frac{d^3\vec{p_2}}{(2\pi)^3 2E_2} \quad \mathbf{(21)}$$

- The integral is Lorentz invariant $\Longrightarrow$ can be evaluated in any frame $\Longrightarrow$ center-of-mass (CM) frame is the most convenient as the mother particle is at rest $\Longrightarrow$ $E_i = m_i, \vec{p_i} = 0$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\delta^3(\vec{p_1} + \vec{p_2}) \frac{d^3\vec{p_1}}{2E_1}\frac{d^3\vec{p_2}}{2E_2} \quad \mathbf{(22)}$$

- Integrating over $\vec{p_2}$ using the delta function

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \frac{d^3\vec{p_1}}{4E_1 E_2} \quad \mathbf{(23)}$$

Decay rate calculation

- The integration over $\overrightarrow{p_2}$ using the δ –function imposes $\overrightarrow{\mathrm{p}_2} = -\overrightarrow{p_1}$ and therefore $E_2^2 = m_2^2 + |\overrightarrow{p_1}|^2$

- We can then write $d^3\overrightarrow{p_1} = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$ which leads to:

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right) \frac{p_1^2 dp_1 d\Omega}{4E_1 E_2} \qquad \mathbf{(24)}$$

$$\Downarrow$$

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 g(p_1) \delta\big(f(p_1)\big) dp_1 d\Omega \qquad \mathbf{(25)}$$

Decay rate calculation

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int \left|M_{fi}\right|^2 g(p_1)\delta\big(f(p_1)\big)\, dp_1 d\Omega \qquad \mathbf{(25)}$$

$$g(p_1) = \frac{p_1^2}{E_1 E_2} = \frac{p_1^2}{\sqrt{(m_1^2 + p_1^2)(m_2^2 + p_1^2)}} \qquad f(p_1) = m_i - \sqrt{(m_1^2 + p_1^2)} - \sqrt{(m_2^2 + p_1^2)} \qquad \mathbf{(26)}$$

- Note that $f(p_1)$ imposes energy conservation!
- CM momenta of the two decay products is fixed by $f(p_1) = 0$ for $p_1 = -p_2 = p^*$

Decay rate calculation

- Integrating Eq. 25 and using the property of the δ −function we get

$$\int g(p_1)\delta\big(f(p_1)\big)dp_1 = \frac{1}{|df/dp_1|_{p^*}}\int g(p_1)\delta(p-p^*)dp_1 = g(p^*)/\left|\frac{df}{dp_1}\right|_{p^*} \qquad \mathbf{(27)}$$

- Here, p^* is the value for which $f(p^*) = 0$

$$\frac{df}{dp_1} = -\frac{p_1}{\sqrt{m_1^2+p_1^2}} - \frac{p_1}{\sqrt{m_1^2+p_1^2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\,\frac{E_1+E_2}{E_1E_2} \qquad \mathbf{(28)}$$

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int \left|M_{fi}\right|^2 \left|\frac{E_1E_2}{p_1(E_1+E_2)}\,\frac{p_1^2}{E_1E_2}\right|_{p_1=p^*} d\Omega = \frac{1}{32\pi^2 E_i}\int \left|M_{fi}\right|^2 \left|\frac{p_1}{(E_1+E_2)}\right|_{p_1=p^*} d\Omega \qquad \mathbf{(29)}$$

- From $f(p_1) = 0$ (energy conservation) we get $m_i = E_1 + E_2$

$$\boldsymbol{\Gamma_{fi} = \frac{|\overrightarrow{p^*}|}{32\pi^2 E_i m_i}\int \left|M_{fi}\right|^2 d\Omega} \qquad \mathbf{(30)}$$

Decay rate calculation

- In the particle`s rest frame: $E_i = m_i$

$$\frac{\mathbf{1}}{\boldsymbol{\tau}} = \boldsymbol{\Gamma} = \frac{\left|\overrightarrow{\boldsymbol{p}^*}\right|}{\mathbf{32}\boldsymbol{\pi}^{\mathbf{2}}\boldsymbol{m}_{\boldsymbol{i}}^{\mathbf{2}}} \int \left|\boldsymbol{M}_{\boldsymbol{fi}}\right|^{\mathbf{2}} \boldsymbol{d\Omega} \qquad (\mathbf{30})$$

- Valid for all two-body decays – fundamental physics contained in the matrix element, additional factors arise from the phase-space integral

- p^* can be obtained from $f(p_1) = 0$

$$m_i = \sqrt{m_1^2 + p^{*2}} + \sqrt{m_2^2 + p^{*2}}$$
$$\Longrightarrow p^* = \frac{1}{2m_i^2}\sqrt{\left[m_i^2 - (m_1 + m_2)^2\right]\left[m_i^2 - (m_1 - m_2)^2\right]} \qquad (\mathbf{31})$$

Particle decays

- A given particle may decay to more than one decay mode

- The total decay rate per unit time Γ is the sum of the individual decay rates

$$\Gamma = \sum_j \Gamma_j \qquad (32)$$

- N remaining particles after time t

$$N(t) = N(0)e^{-\Gamma t} = N(0)e^{-t/\tau} \qquad , \tau = 1/\Gamma \qquad (33)$$

- The branching fraction for a specific decay mode is simply given by:

$$\mathrm{B(j)} = \frac{\Gamma_j}{\Gamma} \qquad (34)$$

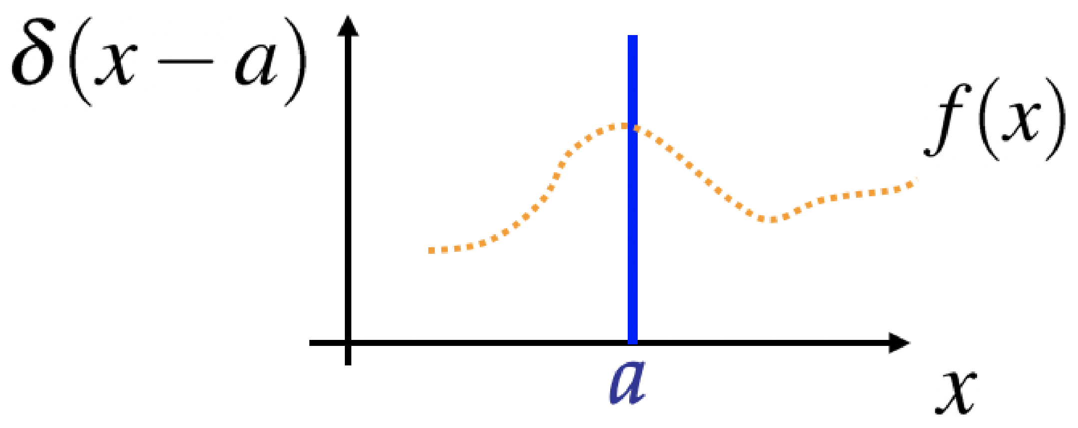
Summary of Lecture 4

Main learning outcomes

- How to compute 2-body decay rates using Fermi's golden rule
- How to deal with kinematics of particle decays
- The fundamental particle physics is in the matrix element
- The above equations are the basis for all calculations that follow

Additional slides: Dirac δ −function

- In the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: “infinitely narrow spike of unit area”



$$\int_{-\infty}^{\infty} \delta(x-a)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

- Any function with the above properties can represent $\delta(x)$, e.g.:

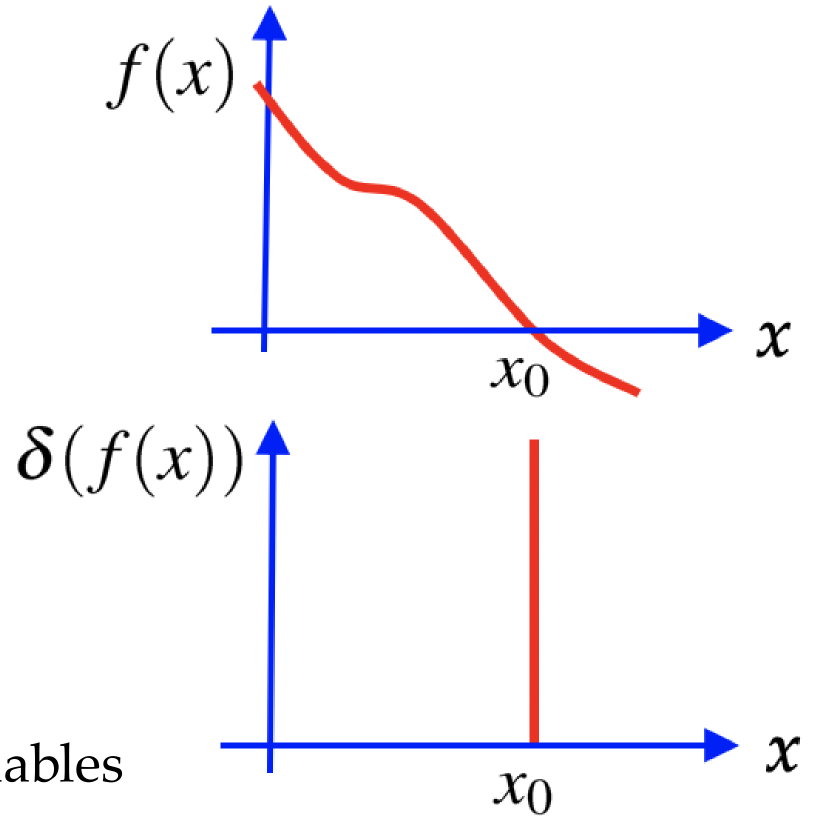
$$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma}} e^{-\left(\frac{x^2}{2\sigma^2}\right)}$$

Infinitesimally narrow Gaussian

Additional slides: Dirac δ −function of a function

- An expression for the δ −function of a function $\delta(f(x))$:
  - start from the definition of a δ −function:

$$\int_{y_1}^{y_2} \delta(y)dy = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$



- Now express in terms of $y = f(x)$, where $f(x_0) = 0$ and change variables

$$\int_{x_1}^{x_2} \delta(f(x)) \frac{df}{dx} dx = \begin{cases} 1 & if\ x_1 < 0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

Additional slides: Dirac δ −function of a function

- From the properties of a δ −function (i.e. only non-zero at x_0)

$$\left|\frac{df}{dx}\right| \int_{x_1}^{x_2} \delta\big(f(x)\big)dx = \begin{cases} 1 & if\ x_1 < 0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- Rearranging and expressing RHS as a δ −function

$$\int_{x_1}^{x_2} \delta\big(f(x)\big)dx = \frac{1}{\left|\frac{df}{dx}\right|_{x_0}} \int_{x_1}^{x_2} \delta(x - x_0)dx \Longrightarrow \delta\big(f(x)\big) = \left|\frac{df}{dx}\right|_{x_0}^{-1} \delta(x - x_0)$$